NPS-MA-91-012

NAVAL POSTGRADUATE SCHOOL

Monterey, California

ELLIPTIC INTEGRALS
IN THE
MINKOWSKI PLANE

by

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Technical Report for Period

October 1990 - May 1991

Prepared for: Naval Postgraduate School
Monterey, CA 93943

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA 93943

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No 0704-0188

1a REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S): NPS-MA-91-012	5 MONITORING ORGANIZATION REPORT NUMBER(S): NPS-MA-91-012
6a NAME OF PERFORMING ORGANIZATION: Naval Postgraduate School; 6b OFFICE SYMBOL (If applicable): MA	7a NAME OF MONITORING ORGANIZATION: Naval Postgraduate School
6c ADDRESS (City, State, and ZIP Code): Monterey, CA 93943	7b ADDRESS (City, State, and ZIP Code): Monterey, CA 93943
8a NAME OF FUNDING/SPONSORING ORGANIZATION: Naval Postgraduate School; 8b OFFICE SYMBOL (If applicable): MA	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: O&MN Direct Funding
8c ADDRESS (City, State, and ZIP Code): Monterey, CA 93943	10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO; PROJECT NO; TASK NO; WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)

ELLIPTIC INTEGRALS IN THE MINKOWSKI PLANE UNCLASSIFIED

12 PERSONAL AUTHOR(S)

Mostafa Ghandehari & Dave Logothetti

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Technical	FROM 10/91 TO 05/91	9 May 1991	14

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Convex sets and related geometric topics, Elliptic integrals, Minkowski plane

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

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20 DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL: Mostafa Ghandehari	22b TELEPHONE (Include Area Code): (408) 646-2124; 22c OFFICE SYMBOL: MA

Elliptic Integrals in the Minkowski Plane

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Abstract

Expressions for the perimeter of the unit circle of a normal linear plane in terms of elliptic integrals of the first and second kinds are given. Asymptotic results, where the center is close to the boundary are discussed. An affine transformation is used to obtain similar results for an ellipse.

1. <u>Introduction</u>. We use elliptic integrals of the first and second kind. Kline [6] contains a historical account of elliptic integrals. Abramowitz and Stegun [2] contains properties and tables for elliptic integrals.

An elliptic integral of the first kind is given by

$$K(r, \varphi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - r^2 \sin^2 \varphi}} \, , 0 < r < 1 \, , \tag{1}$$

while an elliptic integral of the second kind is given by

$$E(r, \varphi) = \int_0^{\varphi} \sqrt{1 - r^2 \sin^2 \varphi} \quad d\varphi, \, 0 < r < 1 \, . \tag{2}$$

If the upper limit $\varphi = \frac{\pi}{2}$ then these integrals are called complete. Otherwise they are called incomplete.

Let K be a centrally symmetric plane convex body, centered at the origin. We can think of K as the unit disk for a 2-dimensional Banach space, or Minkowski plane. Let $\sigma(K)$ denote the length of K, computed using the metric induced by K. Then $\sigma(K)$ is called the "self-circumference" of K. More generally if K is not necessarily centrally symmetric, and z is any interior point of K, we can define $\sigma_+ (K, z)$ and $\sigma_- (K, z)$ the respective positive and negative self-circumferences of K at z, which both reduce to $\sigma(K)$ in case K is centrally symmetric with z as its center. It is shown that if $K = B$, a Euclidean unit circle, and z is at a distance r, $0 < r < 1$, from the center then $\sigma_+ (B, z)$ can be expressed in terms of the complete elliptic integral of the second kind. In the next section we define a related function $\tau(K, z)$ for a convex body K and show that if $K = B$, a Euclidean unit circle, with z a distance $r < 1$ from the center then $\tau(B, z)$ is given by a complete elliptic integral of the first kind.

Section 2 contains preliminary definitions and concepts. Calculation of self-circumference in terms of elliptic integrals is given in section 3. An asymptotic analysis as z approaches the boundary of a unit circle as well as other properties of self-circumference is given in section 4. In section 5 an affine transformation is used to discuss similar results for an ellipse.

2. Preliminaries. By a plane convex body we mean a compact convex subset of the Euclidean plane with nonempty interior. Let K be a plane convex body with the origin as an interior point. For each angle θ, $0 \leq \theta \leq 2\pi$ we let $r(K, \theta)$ be the radius of K in direction θ, so that the boundary of K has equation $r = r(K, \theta)$ in polar coordinates. The distance from the origin to the supporting line of K with outward unit normal $(\cos\theta, \sin\theta)$ is denoted by $h(K, \theta)$. This is the supporting function of K restricted to the unit circle. Since K is convex, it has a well-defined unique tangent line at all but a countable number of points. We let $ds(K, \theta)$ represent the element of Euclidean arclength of the boundary of K at a point where the unit normal is given by

$$L(K) = \int_0^{2\pi} h(K, \theta)\, d\theta . \tag{3}$$

The polar dual of K, denoted by K^*, is another plane convex body having the origin as an interior point and is defined in such a way that

$$h(K^*, \theta) = \frac{1}{r(K, \theta)} \text{ and } r(K^*, \theta) = \frac{1}{h(K, \theta)} . \tag{4}$$

A result which we will use in our later discussion is Steinhardt's inequality [8] given by

$$L(K)\, L(K^*) \geq 4\pi^2 . \tag{5}$$

If K is a centrally symmetric plane convex body centered at the origin, then the self-circumference $\sigma(K)$ is given by

$$\sigma(K) = \int_0^{2\pi} \frac{ds(K, \theta)}{r(K, \theta + \frac{\pi}{2})} . \tag{6}$$

If K is not necessarily symmetric and z is any point interior to K, then positive and negative self-circumference of K relative to z are defined by

$$\sigma_+ (K, z) = \int_0^{2\pi} \frac{ds(K, \theta)}{r(K, \theta + \frac{\pi}{2})} \tag{7}$$

and

$$\sigma_{-}(K, z) = \int_0^{2\pi} \frac{ds(K, \theta)}{r(K, \theta - \frac{\pi}{2})} \tag{8}$$

where the origin of the coordinate system is at z. Both $\sigma_{+}(K, z)$ and $\sigma_{-}(K, z)$ reduce to $\sigma(K)$ in case K is centrally symmetric with z as its center.

If K_1 and K_2 are plane convex bodies with the origin as an interior point, then the length of the positively oriented boundary of K_1, with respect to K_2 is given by

$$\sigma_{+}(K_1, K_2) = \int_0^{2\pi} \frac{ds(K_1, \theta)}{r(K_2, \theta + \frac{\pi}{2})} \tag{9}$$

and the length of the negatively oriented boundary is given by

$$\sigma_{-}(K_1, K_2) = \int_0^{2\pi} \frac{ds(K_1, \theta)}{r(K_2, \theta - \frac{\pi}{2})} . \tag{10}$$

A natural generalization of the function σ is the following function $\tau(K, z)$ for a plane convex body K at a point z defined by

$$\tau(K, z) = \int_0^{2\pi} \frac{ds(K, \theta)}{r(K, \theta) + r(K, \theta + \pi)} . \tag{11}$$

In case K is centrally symmetric with z as the center, then $\tau(K, z)$ represents half of the Minkowskian "self-circumference". More generally if K is a convex body in R^n, $n \geq 2$ then let z be an interior point of K, and for each direction u let $A(u)$ be the area (i.e. (n-1)-dimensional volume) of the intersections of K with the hyperplane passing through z and having unit normal u. Let $d\,S(u)$ be the area element of the boundary of K at a point with outward unit normal u. We define

$$\tau(K, z) = \int \frac{d\,S(u)}{A(u)} , \text{ where} \tag{12}$$

the integration is over the boundary of K. Then $\tau(K, z)$ is affine invariant. Chakerian and Talley [3] investigates properties of τ. In particular it is shown there that if B^n is the unit n-ball in R^n, then

$$\tau(B^n, z) = 2\sqrt{\pi}\left(\Gamma(\tfrac{n}{2})\right)^{-1}\Gamma\left(\frac{n+1}{2}\right)F\left(\frac{n-1}{2}, \frac{1}{2}, \frac{n}{2}, r^2\right) \tag{13}$$

where $r < 1$ is the distance from z to the center of B^n, Γ is the Gamma function, and F is the Gauss hypergeometric function. In the case $n = 2$, (13) reduces to an elliptic integral of the first kind via the relation

$$K(r, \tfrac{\pi}{2}) = \frac{1}{2}\pi F\left(\frac{1}{2}, \frac{1}{2}, 1, r^2\right). \tag{14}$$

We next use definitions and concepts developed above to discuss self-circumference of a Euclidean unit circle with respect to an interior point, not necessarily the center, in terms of elliptic integrals. Asymptotic results as z approaches the boundary are included. An affine transformation will give related results for an ellipse.

3. The unit circle. In this section we calculate functions σ and τ defined earlier for a Euclidean unit circle with respect to an interior point. Thus we view a Euclidean unit circle as a unit circle of a Minkowski plane with a norm which is not necessarily symmetric. The following theorem gives a result for the positive self-circumference a Euclidean unit circle with respect to an interior point.

Theorem 1. Let B denote a Euclidean unit circle. Assume z is an interior point of B with distance $r < 1$ from the center. Then the positive self-circumference $\sigma_+(B, z)$ is given by

$$\sigma_+(B, Z) = \frac{4}{1 - r^2}\int_0^{\frac{\pi}{2}} \sqrt{1 - r^2 \sin^2\theta}\; d\theta = \frac{4}{1 - r^2}\, E\left(r, \frac{\pi}{2}\right). \tag{15}$$

Proof. Without loss of generality assume z is on a horizontal axis as it is shown in Figure 1. Let 0 denote the center of B. Let P denote a point on the boundary of B such that the unit normal is given by $(\cos\theta, \sin\theta)$. Draw a perpendicular to OP from z and assume that it intersects the boundary of B at Q. Using simple trigonometry it follows that

$$d(z, Q) = r\left(B, \theta + \tfrac{\pi}{2}\right) = r \sin\theta + \sqrt{1 - r^2 \cos^2\theta} .$$

Since the arc length element $ds(B, \theta)$ is equal to $d\theta$ for the Euclidean unit circle, by using (7) we obtain

$$\sigma_+(B, z) = \int_0^{2\pi} \frac{d\theta}{r \sin\theta + \sqrt{1 - r^2 \cos^2\theta}} .$$

After rationalizing the denominator we obtain

$$\sigma_+(B, z) = \frac{4}{1 - r^2} \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2 \sin^2\theta} \; d\theta ,$$

which gives the desired result by giving the definition given in (2) . ■

The following theorem gives a similar result to theorem 1 for calculating the function τ defined earlier.

Theorem 2. Consider a Euclidean unit circle B . Let z be an interior point with distance $r < 1$ from the center. Then the self-circumference τ is given by

$$\tau(B, z) = 2\, K(r, \tfrac{\pi}{2}) \tag{16}$$

where K denotes the complete elliptic integral of the first kind.

Proof. The proof follows from the resullt of Chakerian and Talley given in formula (13) and the formula (14) relating Gauss's hypergeometric function and the complete elliptic integral of first kind. A direct calculation similar to the proof of theorem 1 will also give the proof. ■

In the next theorem we use the following Landen's transformation for the hypergeometric function

$$F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1, \frac{4x}{(1 + x)^2}\right) = (1 + x)\, F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1, x^2\right) . \tag{17}$$

See Amkvist and Berndt [1, p 442] for more detail on the above formula. Their article is also very interesting and discusses calculation of π and related topics.

Theorem 3. Consider a Euclidean unit circle as drawn in Figure 2. Assume $r < 1$. Let P denote a point with distance $\sqrt{r}$ from the center 0. Label A and B on a horizontal axis as shown. Connect A to P and extend to intersect the circle at B'. Draw $B' A'$ parallel to BA and let the intersection with a vertical axis he called z^*. Draw a perpendicular to AB' and denote by z the intersection with the horizontal. Denote the unit circle by K. Then the function τ satisfies

$$\tau(K, z^*) = (1 + r)\, \tau(K, z) \tag{18}$$

Proof. Using $n = 2$ and formula (13) we obtain

$$\tau(K, z) = \pi\, F\left(\frac{1}{2}, \frac{1}{2}, 1, r^2\right),$$

where we have used similar triangles to conclude $d(0, z) = r$.
By some simple trigonometry using the formula

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

we obtain

$$d(0, z^*) = \frac{2\sqrt{r}}{1 + r}.$$

By (17) it follows that

$$\pi\, F\left(\frac{1}{2}, \frac{1}{2}, 1, \left(\frac{2\sqrt{r}}{1 + r}\right)^2\right) = (1 + r)\, \pi\, F\left(\frac{1}{2}, \frac{1}{2}, 1, r^2\right).$$

By using theorem 2 we obtain the desired result given in (18). ∎

The following theorem gives the self-circumference of a Euclidean unit circle with respect to a convex curve K. The proof is given by the author in [5] and is omitted.

Theorem 4. Let K be a plane convex body. Assume B is the Euclidean unit circle. Then the length of B with respect to K is equal to the Euclidean length of the polar dual of K. This is

$$(19) \qquad \sigma_+(B, K) = L(K^*) \quad \text{where}$$

K^* denotes the polar dual of K .

Using formulas (3) and (4) and the law of cosine the length of polar duals of convex curves containing circular arcs will also yield elliptic integrals. The calculation for specific curves will be treated in another work.

4. Asymptotic Estimates. In this section asymptotic results for self-circumference of a unit circle where the center is close to the boundary is discussed.

Using the relation between elliptic integrals and the hypergeometric function formulas (15) and (16) given in theorems 1 and 2 reduce to

$$(20) \qquad \sigma_+(B, z) = \frac{2\pi}{1 - r^2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, r^2) \text{ and}$$

$$(21) \qquad \tau(B, z) = \pi F(\tfrac{1}{2}, \tfrac{1}{2}, 1, r^2) .$$

To obtain an asymptotic result for σ_+ and τ given by (20) and (21) we use the following result (22) from Erdélyi et al [4, p 74] .

We first introduce some notation before stating the theorem. Let Γ denote the Gamma function and F the hypergeometric function. Let ψ denote the Euler ψ function. Use the shifted factorial Pochhammer notation

$$(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} .$$

Let $F(a , b , c , \zeta)$ denote the hypergeometric function. Assume $c - a - b = d$, where d is a positive integer. Then in a neighborhood of $\zeta = 1$,

$$(22) \qquad F(a , b , a + b + d , \zeta) = \frac{\Gamma(d)\, \Gamma(a + b + d)}{\Gamma(a + d)\, \Gamma(b + d)} \sum_{n = 0}^{d - 1} \frac{(a)_n \; (b)_n}{(1 - d)_n \; n!} (1 - \zeta)^n$$

$$+(1 - \zeta)^d (- 1)^d \frac{\Gamma(a + b + d)}{\Gamma(a)\, \Gamma(b)} \cdot \sum_{n = 0}^{\infty} \frac{(a + d)_n \; (b + d)_n}{n!\, (n + d)\, !} \cdot$$

$$[k_n - \log (1 - \zeta)] \; (1 - \zeta)^n \quad \text{where}$$

$k_n = \psi(n+1) + \psi(n+1+d) - \psi(a+n+d) - \psi(b+n-d)$ and where $\sum_{n=0}^{d-1}$ denotes zero if $d = 0$.

By using (20), (21) and (22) we obtain the first few terms of asymptotic calculation for σ_+ and τ as follows:

$$\sigma_+(B, z) \sim \frac{4}{1 - r^2} + 4\log 2 - 1 - \log(1 - r^2)\,, \tag{23}$$

$$\tau(B, z) \sim -\log(1 - r^2) + 4\log 2\,. \tag{24}$$

By using (13) and (22) we obtain

$$\tau(B^n, z) \sim -(n-1)\log(1 - r^2)\,. \tag{25}$$

More terms for results in (23), (24) and (25) can be obtained by carrying out more terms of the formula given in (22) for each case.

5. Ellipse. *In this section we treat the* self-circumference of an ellipse. Using the fact that self-circumference is affine invariant we obtain the following theorem 5. The proof follows from theorems 1 and 2 and the affine invariance of the self-circumference.

Theorem 5. Consider an ellipse K with major and minor semi-axes a, b and eccentriaty $e = \frac{c}{a}$ where $c = \sqrt{a^2 - b^2}$. Let P denote one of the foci of the ellipse. Then

$$\sigma_+(K, P) = \frac{2\pi}{1 - e^2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, e^2) \quad \text{and} \tag{26}$$

$$\tau(K, P) = \pi F(\tfrac{1}{2}, \tfrac{1}{2}, 1, e^2)\,. \tag{27}$$

The exact formula for the Euclidean length of an ellipse is given by

$$L(a, b) = 2\pi a F(\tfrac{1}{2}, -\tfrac{1}{2}, 1, e^2)\,. \tag{28}$$

See [1, 598] for more detail on the above formula. Then combining (26) and (28) together with properties of elliptic integrals and the hypergeometric series gives

(29) $$\sigma_+(K, P) = \frac{a\, L(a, b)}{b^2} \quad \text{where}$$

L(a, b) denotes the Euclidean length of an ellipse with semi axes a and b.

Let $L^*(a, b)$ denote the Euclidean length of the polar dual of an ellipse with respect to the center. Then by using (3) and (4) a straight forward calculation yields

(30) $$L^*(a, b) = \frac{4}{a} \int_0^{2\pi} \frac{1}{\sqrt{1 - e^2 \cos^2 t}}\, dt \, .$$

By using (27) and (30) we obtain

(31) $$\tau(K, P) = \frac{a}{\text{[illegible]}} L^*(a, b) \, .$$

After multiplying (29) and (31) and using Steinhardt's inequality by (5) we obtain

(32) $$\sigma_+(K, P) \ \tau(K, P) \geq 2\pi^2$$

with equality if and only if K is a circle. This result could be obtained directly by using the definition of elliptic integrals of the first and second kinds and the Cauchy-Schwarz inequality. Pfiefer [7] discusses inequalities for perimeter of an ellipse. By using bounds for perimeter of an ellipse and the formula (29) inequality for self-circumference of an ellipse can be obtained.

Acknowledgements

The authors are thankful to professor Gordon Latta for useful suggestions related to asymptotic calculations in this article. This work is dedicated to the first author's son Javid for the occasion of his fourteenth birthday.

References

[1] G. Almkvist, B. Berndt, Gauss, Landen, Ramanujan, the Arithmetic-Geometric Mean, Ellipses, π, and the Ladies Diary, Amer. Math. Monthly, August-September 1988, pp 585-608.

[2] M. Abromowitz and I.A. Stegun: Handbook of Mathematical Functions, Dover, New York; NBS Applied Math Series 55, U.S. Dept. of Commerce, Washington, D.C. (1964); Chapter 17.

[3] G.D. Chakerian and W.K. Talley, some properties of the self-circumference of convex sets, Arch. Math. 20 (1969), 431-443.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, Higher Transcendental Functions, vol. 1, McGrow-Hill Book Company, New York, 1953.

[5] M. Ghandehari, Steinhardt's inequality in the Minkowski plane, submitted.

[6] M. Kline, Mathematical Thought from Ancient to Modern Times, Oxford University press, 1972, pp 411-422.

[7] R. Pfiefer, Bounds on the perimeter of an ellipse via Minkowski sums, September 1988, College Math. Journal.

[8] F. Steinhardt, On distance functions and on polar series of convex bodies, PhD Thesis, Columbia Univ. 1951.

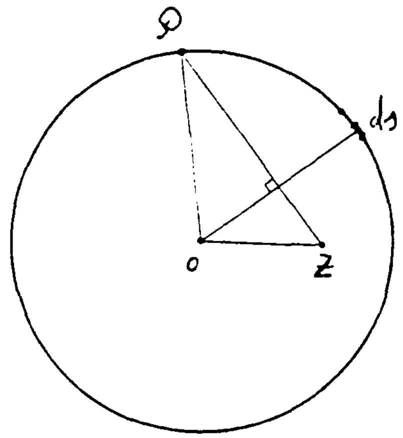

Figure 1

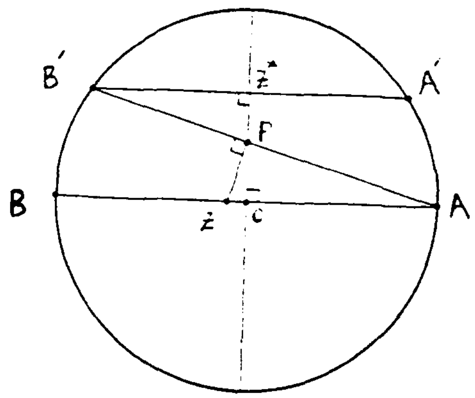

Figure 2